Answers final exam
Quantum Physics I of 28 Nov 2013

Problem 1

a) The eigenvalues can be calculated using

$\hat{H}|\varphi_g\rangle = E_g|\varphi_g\rangle$ and $\hat{H}|\varphi_e\rangle = E_e|\varphi_e\rangle$

E_g: $\begin{pmatrix} E_0 & T \\ T & E_0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(E_0+T) \\ \frac{1}{\sqrt{2}}(E_0+T) \end{pmatrix} = (E_0+T)\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \Rightarrow$

$$E_g = E_0 + T = E_0 - |T|$$

E_e: $\begin{pmatrix} E_0 & T \\ T & E_0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(E_0-T) \\ \frac{-1}{\sqrt{2}}(E_0-T) \end{pmatrix} = (E_0-T)\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \Rightarrow$

$$E_e = E_0 - T = E_0 + |T|$$

b) $\langle\varphi_L|\hat{H}|\varphi_L\rangle = (1\ 0)\begin{pmatrix} E_0 & T \\ T & E_0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = E_0$

The expectation value for total energy when the particle is in state $|\varphi_L\rangle$

$\langle\varphi_R|\hat{H}|\varphi_R\rangle = (0\ 1)\begin{pmatrix} E_0 & T \\ T & E_0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = E_0$

The same, for the state $|\varphi_R\rangle$

$\langle\varphi_L|\hat{H}|\varphi_R\rangle = (1\ 0)\begin{pmatrix} E_0 & T \\ T & E_0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = T$

The energy scale associated with the tunnel coupling between the states $|\varphi_L\rangle$ and $|\varphi_R\rangle$.

$\langle\varphi_R|\hat{H}|\varphi_L\rangle = (0\ 1)\begin{pmatrix} E_0 & T \\ T & E_0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = T$ $\Rightarrow$ Same meaning as $\langle\varphi_L|\hat{H}|\varphi_R\rangle$.

c) $\langle\varphi_g|\hat{H}|\varphi_g\rangle = \langle\varphi_g|E_g|\varphi_g\rangle = E_g$ (using a))

$\langle\varphi_e|\hat{H}|\varphi_e\rangle = \langle\varphi_e|E_e|\varphi_e\rangle = E_e$ (using a))

These are the expectation values and also eigenvalues for total energy for the system in the state $|\varphi_g\rangle$ and $|\varphi_e\rangle$.

$\langle\varphi_g|\hat{H}|\varphi_e\rangle = \langle\varphi_e|\hat{H}|\varphi_g\rangle = 0$,

since the eigenstates are orthogonal.

This gives that there is no further meaning that the states are orthogonal, no coupling = zero coupling energy between $|\varphi_g\rangle$ and $|\varphi_e\rangle$.

d) The state $\alpha|\varphi_g\rangle + \beta|\varphi_e\rangle$ is normalized for $|\alpha|^2 + |\beta|^2 = 1$. For $|\psi_0\rangle$ this gives $|3|^2 + |e^{i\gamma}|^2 = |3|^2 + |1|^2 = 10$. The normalized version $|\psi_{0N}\rangle$ of $|\psi_0\rangle$ is thus $|\psi_{0N}\rangle = \frac{1}{\sqrt{10}}(3|\varphi_g\rangle + e^{i\gamma}|\varphi_e\rangle)$.

Now calculate $\langle\hat{A}\rangle = \langle\psi_{0N}|\hat{A}|\psi_{0N}\rangle$

$$= \frac{1}{10}\left(\frac{3}{\sqrt{2}} + \frac{e^{-i\gamma}}{\sqrt{2}},\ \frac{3}{\sqrt{2}} - \frac{e^{-i\gamma}}{\sqrt{2}}\right)\begin{pmatrix} -a & 0 \\ 0 & +a \end{pmatrix}\begin{pmatrix} \frac{3}{\sqrt{2}} + \frac{e^{+i\gamma}}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} - \frac{e^{+i\gamma}}{\sqrt{2}} \end{pmatrix} =$$

$$= \frac{a}{10}\left(\frac{3}{\sqrt{2}} + \frac{e^{-i\gamma}}{\sqrt{2}},\ \frac{3}{\sqrt{2}} - \frac{e^{-i\gamma}}{\sqrt{2}}\right)\begin{pmatrix} -\frac{3}{\sqrt{2}} - \frac{e^{+i\gamma}}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} - \frac{e^{+i\gamma}}{\sqrt{2}} \end{pmatrix}$$

$$= \frac{a}{10}\left(\left(\frac{3}{\sqrt{2}} + \frac{e^{-i\gamma}}{\sqrt{2}}\right)\left(\frac{-3}{\sqrt{2}} - \frac{e^{+i\gamma}}{\sqrt{2}}\right) + \left(\frac{3}{\sqrt{2}} - \frac{e^{-i\gamma}}{\sqrt{2}}\right)\left(\frac{3}{\sqrt{2}} - \frac{e^{+i\gamma}}{\sqrt{2}}\right)\right)$$

$$= \frac{a}{10}\left(-\frac{9}{2} - \frac{1}{2} + \frac{9}{2} + \frac{1}{2} + \frac{1}{2}\left(-3e^{+i\gamma} - 3e^{-i\gamma} - 3e^{+i\gamma} - 3e^{-i\gamma}\right)\right)$$

3/10

$$= \frac{-6a}{20}\left(e^{+i\gamma} + e^{-i\gamma}\right)$$

$$= -\frac{12}{20} a \cos\gamma = -\frac{3}{5}\cos(\gamma)\cdot a$$

e) $$|\psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|\psi(t_{=0})\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|\varphi_L\rangle$$

$$= \frac{1}{\sqrt{2}}\left(e^{-\frac{i}{\hbar}E_g t}|\varphi_g\rangle + e^{-\frac{i}{\hbar}E_e t}|\varphi_e\rangle\right)$$

e1) $$\langle \hat{H}(t)\rangle = \langle\psi(t)|\hat{H}|\psi(t)\rangle = \frac{1}{2}\left(e^{+\frac{i}{\hbar}E_g t}\langle\varphi_g| + e^{+\frac{i}{\hbar}E_e t}\langle\varphi_e|\right)\hat{H}\left(e^{-\frac{i}{\hbar}E_g t}|\varphi_g\rangle + e^{-\frac{i}{\hbar}E_e t}|\varphi_e\rangle\right)$$

$\boxed{\text{use } \langle\varphi_g|\hat{H}|\varphi_e\rangle = 0}$

$$= \frac{1}{2}\left(\langle\varphi_g|\hat{H}|\varphi_g\rangle + \langle\varphi_e|\hat{H}|\varphi_e\rangle\right) = \frac{1}{2}(E_g + E_e) = E_0$$

e2) Matrix for potential energy is $\hat{V} \Leftrightarrow \begin{pmatrix} E_0 & 0 \\ 0 & E_0 \end{pmatrix} \Rightarrow$

$$\langle\hat{V}(t)\rangle = \langle\psi(t)|\hat{V}|\psi(t)\rangle = \frac{1}{2}\left(e^{+\frac{i}{\hbar}E_g t}\langle\varphi_g| + e^{+\frac{i}{\hbar}E_e t}\langle\varphi_e|\right)\hat{V}\left(e^{-\frac{i}{\hbar}E_g t}|\varphi_g\rangle + e^{-\frac{i}{\hbar}E_e t}|\varphi_e\rangle\right)$$

As intermediate step calculate $\langle\varphi_g|\hat{V}|\varphi_g\rangle, \langle\varphi_e|\hat{V}|\varphi_e\rangle, \langle\varphi_g|\hat{V}|\varphi_e\rangle, \langle\varphi_e|\hat{V}|\varphi_g\rangle$

$$\left.\begin{aligned}
\langle\varphi_g|\hat{V}|\varphi_g\rangle &= \left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right)\begin{pmatrix} E_0 & 0 \\ 0 & E_0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = E_0 \\
\langle\varphi_e|\hat{V}|\varphi_e\rangle &= \left(\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\right)\begin{pmatrix} E_0 & 0 \\ 0 & E_0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = E_0 \\
\langle\varphi_g|\hat{V}|\varphi_e\rangle &= \left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right)\begin{pmatrix} E_0 & 0 \\ 0 & E_0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = 0 \\
\langle\varphi_e|\hat{V}|\varphi_g\rangle &= \left(\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\right)\begin{pmatrix} E_0 & 0 \\ 0 & E_0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = 0
\end{aligned}\right\} \Rightarrow$$

$$\langle V(t)\rangle = \frac{1}{2}\left(\langle\varphi_g|\hat{V}|\varphi_g\rangle + \langle\varphi_e|\hat{V}|\varphi_e\rangle\right) = \frac{1}{2}(E_0 + E_0) = E_0$$

4/10

Problem 2

a) $\hat{X}\,\delta(x - x_n) = x_n\,\delta(x - x_n)$ OR

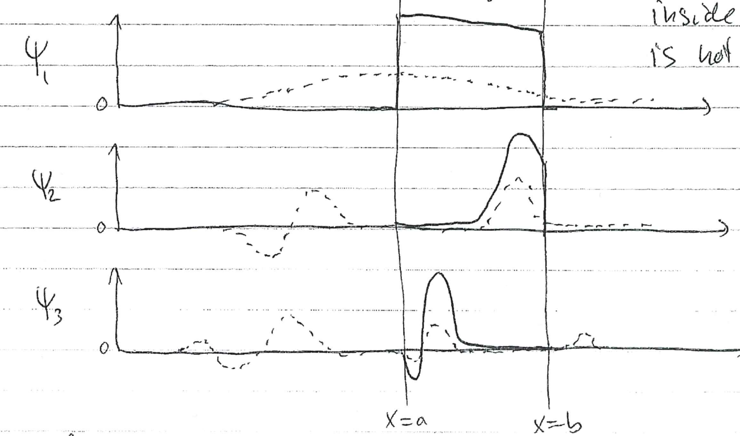
$\hat{X}|x_n\rangle = x_n|x_n\rangle$

$\hat{X}$ is the position operator.
$\delta(x - x_n)$ or $|x_n\rangle$ denotes the eigenstate for position eigenvalue x_n.

b) The position eigenstate is a Dirac delta function located at the position of the eigenvalue, which then is x_n.
Such a state can never be realized in practice, because it would have uncertainty $\Delta X \to 0$. This would required an uncertainty in momentum $\Delta P_x \to \infty$, which costs an infinite amount of energy.
So, it is impossible.

c)

Part of state inside the slit is not disturbed.

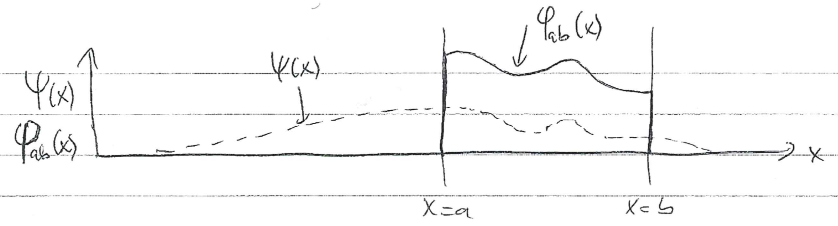


- - - - before measurement

—— after measurement (higher amplitude to maintain normalization)

5/10

d)



State before measurement is $\psi(x)$

State after measurement is $\varphi_{ab}(x) = \begin{cases} 0 & \text{for } x<a,\ x>b \\ C\cdot\psi(x) & \text{for } a\le x\le b \end{cases}$

where C is a constant that accounts for the normalization.

$$\int_a^b |\varphi_{ab}(x)|^2 dx = \int_a^b |C\psi(x)|^2\, dx = 1 \Rightarrow \int_a^b |\psi(x)|^2 dx = \frac{1}{C^2}$$

$$P_{ab} = |\langle \varphi_{ab}|\psi\rangle|^2 = \left|\int_a^b \varphi_{ab}^*(x)\,\psi(x)\,dx\right|^2$$

$$= \left|\int_a^b C\,|\psi(x)|^2\,dx\right|^2 = |C|^2 \left|\int_a^b |\psi(x)|^2 dx\right|^2 = |C|^2 P_{ab}^2$$

But also

$$P_{ab} = \int_a^b |\psi(x)|^2 dx = \frac{1}{C^2} \Rightarrow$$

$$P_{ab} = |C|^2 P_{ab}^2 = \frac{1}{P_{ab}}\cdot P_{ab}^2 = P_{ab} \Rightarrow P_{ab} = P_{ab}$$

Q.E.D.

6/10

Problem 3

a) $\hat{S}_z \Leftrightarrow \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\qquad \hat{S}_y \Leftrightarrow \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$\hat{S}_x \Leftrightarrow \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\hat{H} = -\gamma B_z \hat{S}_z \Leftrightarrow -\gamma B_z \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -\gamma B_z \frac{\hbar}{2} & 0 \\ 0 & +\gamma B_z \frac{\hbar}{2} \end{pmatrix}$$

b) $\hat{S}_x \Leftrightarrow \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ To find the eigen values

Solve $\begin{vmatrix} (0-\lambda) & \frac{\hbar}{2} \\ \frac{\hbar}{2} & (0-\lambda) \end{vmatrix} = 0 \Rightarrow \lambda^2 - \left(\frac{\hbar}{2}\right)^2 = 0 \Rightarrow$

$\lambda = +\frac{\hbar}{2}$ or $\lambda = -\frac{\hbar}{2}$ $\Leftarrow$ The two eigen values

To now find the eigen vectors solve

$$\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \left(\pm\frac{\hbar}{2}\right)\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \frac{\hbar}{2}\begin{pmatrix} b \\ a \end{pmatrix} = \pm\frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow$$

$a = b$ for ~~both~~ eigenvalues, and $|a| = |b| = \frac{1}{\sqrt{2}}$ for normalization.

For eigenvalue $+\frac{\hbar}{2}$, $\frac{\hbar}{2}\begin{pmatrix} b \\ a \end{pmatrix} = +\frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow$ the eigen state is $|+_x\rangle \Leftrightarrow \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

For eigen value $-\frac{\hbar}{2}$, $\frac{\hbar}{2}\begin{pmatrix} b \\ a \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow$ the eigenstate is $|-_x\rangle \Leftrightarrow \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

7/10

c) $\Delta S_z = \sqrt{\langle \hat{S}_z^2 \rangle - \langle S_z \rangle^2}$

$\langle \hat{S}_z \rangle = \left(\sqrt{\tfrac{1}{2}}, +i\sqrt{\tfrac{1}{2}}\right)\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \sqrt{\frac{1}{2}} \\ -i\sqrt{\frac{1}{2}} \end{pmatrix} = 0\,\hbar$

$\hat{S}_z^2 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{\hbar^2}{2^2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\frac{\hbar^2}{4}$ $\Rightarrow$

$\langle \hat{S}_z^2 \rangle = \left(\sqrt{\tfrac{1}{2}}, +i\sqrt{\tfrac{1}{2}}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{\frac{1}{2}} \\ -i\sqrt{\frac{1}{2}} \end{pmatrix}\frac{\hbar^2}{4} = \frac{\hbar^2}{4}$

$\Delta S_z = \sqrt{\frac{\hbar^2}{4} - 0} = \frac{\hbar}{2}$

d) $\langle \uparrow | \hat{S}_x | \uparrow \rangle = (1,0)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\frac{\hbar}{2} = 0$

$\langle \downarrow | \hat{S}_x | \downarrow \rangle = (0,1)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\frac{\hbar}{2} = 0$

$\langle \uparrow | \hat{S}_x | \downarrow \rangle = (1,0)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\frac{\hbar}{2} = +\frac{\hbar}{2}$

$\langle \downarrow | S_x | \uparrow \rangle = (0,1)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\frac{\hbar}{2} = +\frac{\hbar}{2}$

$\langle \uparrow | \hat{S}_z | \uparrow \rangle = (1,0)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\frac{\hbar}{2} = +\frac{\hbar}{2}$

$\langle \downarrow | \hat{S}_z | \downarrow \rangle = (0,1)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\frac{\hbar}{2} = -\frac{\hbar}{2}$

$\langle \uparrow | \hat{S}_z | \downarrow \rangle = (1,0)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\frac{\hbar}{2} = 0$

$\langle \downarrow | \hat{S}_z | \uparrow \rangle = (0,1)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\frac{\hbar}{2} = 0$

8/10

e) $|\psi_0\rangle = \sqrt{\tfrac{1}{2}}\,|\uparrow\rangle - i\sqrt{\tfrac{1}{2}}\,|\downarrow\rangle \Rightarrow$

$|\psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|\psi_0\rangle = \sqrt{\tfrac{1}{2}}\,e^{-i\omega_\uparrow t}|\uparrow\rangle - i\sqrt{\tfrac{1}{2}}\,e^{-i\omega_\downarrow t}|\downarrow\rangle$

with $\omega_\uparrow = \frac{-\gamma B_z}{2}$ and $\omega_\downarrow = \frac{+\gamma B_z}{2}$

$\langle \hat{S}_x \rangle(t) = \langle \psi(t) | \hat{S}_x | \psi(t) \rangle$

$= \frac{1}{2}\left(e^{+i\omega_\uparrow t}\langle\uparrow| + i e^{+i\omega_\downarrow t}\langle\downarrow|\right)\hat{S}_x\left(e^{-i\omega_\uparrow t}|\uparrow\rangle - i e^{-i\omega_\downarrow t}|\downarrow\rangle\right)$

$= \frac{1}{2}\langle\uparrow|S_x|\uparrow\rangle + \frac{1}{2}\langle\downarrow|S_x|\downarrow\rangle + \frac{-i}{2}e^{+i(\omega_\uparrow-\omega_\downarrow)t}\langle\uparrow|\hat{S}_x|\downarrow\rangle + \frac{i}{2}e^{+i(\omega_\downarrow-\omega_\uparrow)t}\langle\downarrow|\hat{S}_x|\uparrow\rangle$

$= \frac{\hbar}{2}\left(-\frac{i}{2}e^{-i(\omega_\downarrow-\omega_\uparrow)t} + \frac{i}{2}e^{+i(\omega_\downarrow-\omega_\uparrow)t}\right)$

$= \frac{\hbar}{4}\left(-i\left(\cancel{\cos(\gamma B_z t)} - i\sin(\gamma B_z t)\right) + i\left(\cancel{\cos(\gamma B_z t)} + i\sin(\gamma B_z t)\right)\right)$

$= \frac{\hbar}{4}\left(-2\sin(\gamma B_z t)\right) = -\frac{\hbar}{2}\sin(\gamma B_z t)$

Problem 4

9/10

a) $\widehat{|L|}\,|l, m_l\rangle = \hbar\sqrt{l(l+1)}\,|l, m_l\rangle$ (length)

$\hat{L}_z\,|l, m_l\rangle = m_l\hbar\,|l, m_l\rangle$ (z-comp.)

$\widehat{|F|}\,|f, m_f\rangle = \hbar\sqrt{f(f+1)}\,|f, m_f\rangle$ (length)

$\vec{F}_z\,|f, m_f\rangle = m_f\hbar\,|f, m_f\rangle$ (z-comp.)

b) Use the rules of addition of angular momentum:

$\vec{F} = \vec{L} + \vec{J}$ with

$f = |l-j|,\ |l-j|+1, \ldots\ |l+j|-1,\ |l+j|$

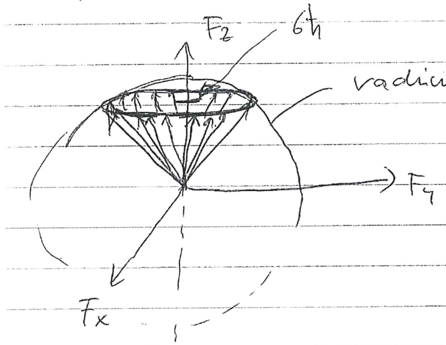
$m_f = -f,\ -(f-1),\ -(f-2) \ldots\ +(f-1),\ +f$

With the equations of a), this gives the maximum $m_f = 6$, so, the maximum $f = 6$, so $|l+j| = 6$. From the remark that $|\vec{L}| = \sqrt{12}\,\hbar$ we know $l(l+1) = 12 \Rightarrow l = 3 \Rightarrow j = 3$

c) c1) After the measurement the state of the system is $|\psi\rangle = |f=6, m_f=6\rangle$, given the reasoning of question b).

10/10

c2) The state $|f=6, m_f=6\rangle$ can be drawn as



radius $\sqrt{f(f+1)}\,\hbar = \sqrt{6\cdot 7}\,\hbar$

(as Fig 4.9 in the Griffiths book) p.165

This state is symmetric around the F_z-axis

$\Rightarrow \langle F_x\rangle = 0\,\hbar$

c3) Use the uncertainty relation

$\Delta F_x \cdot \Delta F_y \gtrsim \frac{\hbar}{2}|\langle F_z\rangle| = \frac{\hbar}{2}\cdot 6\hbar = 3\hbar^2 \Rightarrow$

Given the symmetry in the figure $\Delta F_x \approx \Delta F_y \Rightarrow$

$\Delta F_x = \Delta F_y \approx \sqrt{3}\,\hbar$

there is no reason for ΔF_x and ΔF_y to be much larger.

d) $j=3$ and $l=3$, so $f = 0, 1, 2, 3, 4, 5$ or 6

$|j-l|$ $\qquad$ $|j+l|$

Measurement outcomes of $|\vec{F}|$ can then yield the values $\sqrt{f(f+1)} \Rightarrow$ The possible values are

$0\,\hbar,\ \sqrt{2}\,\hbar,\ \sqrt{6}\,\hbar,\ \sqrt{12}\,\hbar,\ \sqrt{20}\,\hbar,\ \sqrt{30}\,\hbar,\ \sqrt{42}\,\hbar$